

Bézier curves based on Lupaş (p, q) -analogue of Bernstein polynomials in CAGD

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Abstract

In this paper, we use the blending functions of Lupaş type (rational) (p, q) -Bernstein operators based on (p, q) -integers for construction of Lupaş (p, q) -Bézier curves (rational curves) and surfaces (rational surfaces) with shape parameters. We study the nature of degree elevation and degree reduction for Lupaş (p, q) -Bézier Bernstein functions. Parametric curves are represented using Lupaş (p, q) -Bernstein basis.

We introduce affine de Casteljau algorithm for Lupaş type (p, q) -Bernstein Bézier curves. The new curves have some properties similar to q -Bézier curves. Moreover, we construct the corresponding tensor product surfaces over the rectangular domain $(u, v) \in [0, 1] \times [0, 1]$ depending on four parameters. We also study the de Casteljau algorithm and degree evaluation properties of the surfaces for these generalization over the rectangular domain. Furthermore, some fundamental properties for Lupaş type (p, q) -Bernstein Bézier curves and surfaces are discussed. We get q -Bézier surfaces for $(u, v) \in [0, 1] \times [0, 1]$ when we set the parameter $p_1 = p_2 = 1$. In comparison to q -Bézier curves and surfaces based on Lupaş q -Bernstein polynomials, our generalization gives us more flexibility in controlling the shapes of curves and surfaces.

We also show that the (p, q) -analogue of Lupaş Bernstein operator sequence $L_{p_n, q_n}^n(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim_{n \rightarrow \infty} q_n = 1$, $\lim_{n \rightarrow \infty} p_n = 1$ and $\lim_{n \rightarrow \infty} p_n^n = a$, $\lim_{n \rightarrow \infty} q_n^n = b$ with $0 < a, b \leq 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L_{p, q}^n(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Keywords and phrases: (p, q) -integers; (p, q) -analogue of Lupaş Bernstein operators; Limit (p, q) -Lupaş operators; Lupaş (p, q) -Bézier curves and surfaces; de Casteljau algorithm; Tensor product.

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1 Introduction and preliminaries

In 1912, S.N. Bernstein [1] introduced his famous operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

and named it Bernstein polynomials to prove the Weierstrass theorem [7].

In computer aided geometric design (CAGD), basis of Bernstein polynomials plays a significant role in order to preserve the shape of the curves or surfaces. The classical Bézier curves [2] constructed with Bernstein basis functions are one of the most important curves in CAGD [25]. Apart from this, Bernstein polynomials has several applications in approximation theory [7], geometry and computer science due to its fine properties of approximation [18].

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in [4, 18, 24].

The rapid development of q -calculus [26] has led to the discovery of new generalizations of Bernstein polynomials involving q -integers [8, 16, 18, 22].

In 1987, Lupas [8] introduced the first q -analogue of Bernstein operator as follows

$$L_{n,q}(f; x) = \sum_{k=0}^n \frac{f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=1}^n \{(1-x) + q^{j-1}x\}}, \quad (1.2)$$

and investigated its approximating and shape-preserving properties.

In 1996, Phillips [20] proposed another q -variant of the classical Bernstein operator, the so-called Phillips q -Bernstein operators and attracted lots of investigations.

$$B_{n,q}(f; x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1] \quad (1.3)$$

where $B_{n,q} : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and any function $f \in C[0, 1]$.

The q -variants of Bernstein polynomials provide one shape parameter for constructing free-form curves and surfaces, Phillips q -Bernstein operator was applied well in this area. In 2003, Oruç and Phillips [18] used the basis functions of Phillips q -Bernstein operator for construction of q -Bézier curves and studied the properties of degree reduction and elevation.

Recently, Mursaleen et al. [11] applied first the concept of (p, q) -calculus in approximation theory and introduced (p, q) -analogue of Bernstein operators based on (p, q) -integers. They also introduced and studied approximation properties based on (p, q) -integers for (p, q) -analogue of Bernstein-Stancu operators, (p, q) -analogue of Bernstein-Kantorovich, (p, q) -analogue of Bernstein-Shurer operators, (p, q) -analogue of Bleimann-Butzer-Hahn operators and (p, q) -analogue of Lorentz polynomials on a compact disk in [12, 13, 14, 15].

Let us recall certain notations of (p, q) -calculus .

For any $p > 0$ and $q > 0$, the (p, q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \dots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases}$$

where $[n]_q$ denotes the q -integers and $n = 0, 1, 2, \dots$.

The formula for (p, q) -binomial expansion is as follows:

$$\begin{aligned}(ax + by)_{p,q}^n &:= \sum_{k=0}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k, \\ (x + y)_{p,q}^n &= (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y), \\ (1 - x)_{p,q}^n &= (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x),\end{aligned}$$

where (p, q) -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}.$$

Details on (p, q) -calculus can be found in [6, 11, 5].

The (p, q) -Bernstein Operators introduced by Mursaleen et al. [11] is as follows:

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n} [n]_{p,q}}\right), \quad x \in [0, 1]. \quad (1.4)$$

Note when $p = 1$, (p, q) -Bernstein Operators given by 1.4 turns out to be q -Bernstein Operators. Also, we have (p, q) -analogue of Euler's identity as:

$$\begin{aligned}(1 - x)_{p,q}^n &= \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x) \\ &= \sum_{k=0}^n (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k\end{aligned}$$

Another needed formulae, which can be easily derived from Euler's identity for $|\frac{q}{p}| < 1$ is:

$$\sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{(p - q)^k [k]_{p,q}!} = \prod_{k=0}^{\infty} \left\{ 1 + \left(\frac{q}{p}\right)^{j-1} x \right\} \quad (1.5)$$

Again by some simple calculations and using the property of (p, q) -integers, we get (p, q) -analogue of Pascal's relation as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \quad (1.6)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \quad (1.7)$$

Motivated by the idea of (p, q) -calculus and its importance in the field of approximation theory given by the Mursaleen et al., we construct Lupaş type (rational) (p, q) -Bézier curves and surfaces based on (p, q) -integers which is further generalization of q -Bézier curves and surfaces [4, 18, 22, 23].

In next section, We present a new analogue, i.e, Lupaş type (p, q) -analogue of the Bernstein functions.

2 Construction of Lupaş (p, q) -analogue of the Bernstein functions

We set

$$b_{p,q}^{k,n}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}, \quad (2.1)$$

and $b_{p,q}^{0,n}(t), b_{p,q}^{1,n}(t), \dots, b_{p,q}^{n,n}(t)$ are the (p, q) -analogue of the Lupaş q -Bernstein functions [4] of degree n on the interval $[0, 1]$.

Also for substitution $u = \frac{t}{1-t}$ where $t \in [0, 1]$ and $u \in [0, \infty)$, and using Euler's identity for $|\frac{q}{p}| < 1$ is:

$$\sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{(p-q)^k [k]_{p,q}!} = \prod_{k=0}^{\infty} \left\{ 1 + \left(\frac{q}{p} \right)^{j-1} x \right\} \quad (2.2)$$

We define

$$b_{p,q}^{k,\infty}(u) = \frac{q^{\frac{k(k-1)}{2}} u^k}{(p-q)^k [k]_{p,q}! \prod_{j=0}^{\infty} \left\{ 1 + \left(\frac{q}{p} \right)^{j-1} u \right\}} \quad (2.3)$$

where $b_{p,q}^{k,\infty}(u)$ are the (p, q) -analogue of limit Lupaş q -analogue of the Bernstein functions [19] of degree n .

Theorem 2.1 *The Lupaş (p, q) -analogue of the Bernstein functions possess the following properties:*

(1.) *Non-negativity:* $b_{p,q}^{k,n}(t) \geq 0$, $k = 0, 1, \dots, n$, $t \in [0, 1]$.

(2.) *Partition of unity:*

$$\sum_{k=0}^n b_{p,q}^{k,n}(t) = 1, \quad t \in [0, 1].$$

(3.) *End-point property:*

$$b_{p,q}^{k,n}(0) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$b_{p,q}^{k,n}(1) = \begin{cases} 1, & \text{if } k = n \\ 0, & k \neq n \end{cases}$$

(4.) *(p, q) inverse symmetry:*

$$b_{p,q}^{n-k,n}(t) = b_{\frac{1}{p}, \frac{1}{q}}^{k,n}(1-t) = b_{\frac{1}{q}, \frac{1}{p}}^{n-k,n}(t)$$

for $k = 0, 1, \dots, n$.

(5.) *Reducibility: when $p = 1$, formula 2.1 reduces to the Lupaş q -Bernstein bases.*

Note: From Euler's identity for $|\frac{q}{p}| < 1$, we have:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{(p-q)^k [k]_{p,q}!} &= \prod_{k=0}^{\infty} \left\{ 1 + \left(\frac{q}{p} \right)^{j-1} x \right\} \\ \implies \sum_{k=0}^{\infty} b_{p,q}^{k,n}(t) &= 1, \quad t \in [0, 1). \end{aligned} \quad (2.4)$$

Proof:

Properties 1, 3 and 5 are obvious. Here we only give the proofs of properties 2 and 4.

Property 2:

When $t = 1$, the conclusion is clear; when $t \neq 1$, we apply the (p, q) analogue of Newtons Binomial formula:

Consider (2)

$$\begin{aligned} &\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k} \\ &= \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} (1-t)^n \left(\frac{t}{1-t} \right)^k \\ &= \left(p(1-t) + qt \right) \left(p^2(1-t) + q^2t \right) \dots \dots \dots \left(p^{n-1}(1-t) + q^{n-1}t \right) \\ &= \prod_{s=1}^n \left(p^{s-1}(1-t) + q^{s-1}t \right). \end{aligned}$$

Hence

$$\sum_{k=0}^n b_{p,q}^{k,n}(t) = 1$$

Property (4) To prove this result, we need following relations:

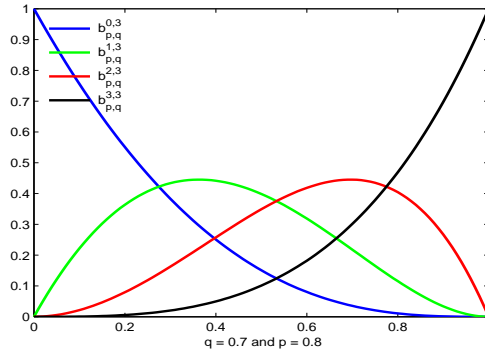
$$[n]_{p,q} = [n]_{q,p} \text{ and } \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} = \left[\begin{matrix} n \\ k \end{matrix} \right]_{\frac{1}{q}, \frac{1}{p}} \frac{(pq)^{\frac{k(2n-1-k)}{2}}}{(pq)^{\frac{k(k-1)}{2}}}.$$

Consider

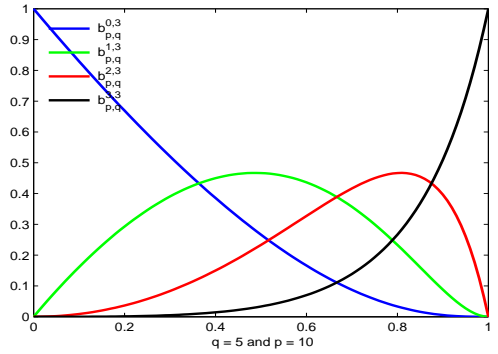
$$\begin{aligned}
b_{p,q}^{n-k,n}(t) &= \frac{\begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q} p^{\frac{(k)(k-1)}{2}} q^{\frac{(n-k)(n-k-1)}{2}} t^{n-k} (1-t)^k}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \\
&= \frac{\begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q} p^{\frac{(k)(k-1)}{2}} q^{\frac{(n-k)(n-k-1)}{2}} t^{n-k} (1-t)^k}{p^{\frac{(n)(n-1)}{2}} q^{\frac{(n)(n-1)}{2}} \prod_{j=1}^n \left\{ \frac{1}{p^{j-1}} t + \frac{1}{q^{j-1}} (1-t) \right\}} \\
&= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}, \frac{1}{p}} \frac{1}{p}^{\frac{(n-k)(n-k-1)}{2}} \frac{1}{q}^{\frac{(k)(k-1)}{2}} t^{n-k} (1-t)^k}{\prod_{j=1}^n \left\{ \frac{1}{p^{j-1}} t + \frac{1}{q^{j-1}} (1-t) \right\}} \\
&= b_{\frac{1}{p}, \frac{1}{q}}^{k,n} (1-t) \\
&= b_{\frac{1}{q}, \frac{1}{p}}^{n-k,n} (t).
\end{aligned}$$

The Lupaş (p, q) -Bernstein blending functions for $n = 3$ are as follows:

$$\begin{aligned}
b_{p,q}^{0,3} &= \frac{p^3(1-t)^3}{(p(1-t) + qt) (p^2(1-t) + q^2t)} \\
b_{p,q}^{1,3} &= \frac{(p^2 + pq + q^2) pt(1-t)^2}{(p(1-t) + qt) (p^2(1-t) + q^2t)} \\
b_{p,q}^{2,3} &= \frac{(p^2 + pq + q^2) qt^2(1-t)}{((p(1-t) + qt) (p^2(1-t) + q^2t))} \\
b_{p,q}^{3,3} &= \frac{q^3t^3}{(p(1-t) + qt) (p^2(1-t) + q^2t)}
\end{aligned}$$



(a) $q = 0.7, p = 0.8$



(b) $q = 5, p = 10$

Figure 1: Lupaş cubic Bézier blending functions

Figure 1a and 1b show the Lupaş (p, q) -Bernstein blending functions of degree 3 for different values of p and q . Here we can observe that sum of blending functions is always unity.

3 Degree elevation and reduction for Lupaş (p, q) -Bernstein functions

Technique of degree elevation has been used to increase the flexibility of a given curve. A degree elevation algorithm calculates a new set of control points by choosing a convex combination of the old set of control points which retains the old end points. For this purpose, the identities (3.1), (3.2) and Theorem (3.1) are useful.

Degree elevation

$$\frac{q^n t}{p^n(1-t) + q^n t} b_{p,q}^{k,n}(t) = \left(1 - \frac{p^{k+1}[n-k]}{n+1}\right) b_{p,q}^{k+1,n+1}(t) \quad (3.1)$$

$$\frac{p^n(1-t)}{p^n(1-t) + q^n t} b_{p,q}^{k,n}(t) = \left(\frac{p^k[n+1-k]}{n+1}\right) b_{p,q}^{k,n+1}(t) \quad (3.2)$$

Theorem 3.1 *Each Lupaş (p, q) -analogue of the corresponding Bernstein function of degree n is a linear combination of two Lupaş (p, q) -analogues of the Bernstein functions of degree $n+1$:*

$$b_{p,q}^{k,n}(t) = \left(\frac{p^k[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) b_{p,q}^{k,n+1}(t) + \left(1 - \frac{p^{k+1}[n-k]_{p,q}}{[n+1]_{p,q}}\right) b_{p,q}^{k+1,n+1}(t) \quad (3.3)$$

Proof:

$$\begin{aligned} b_{p,q}^{k,n}(t) &= b_{p,q}^{k,n}(t) \left(1 - \frac{q^n t}{p^n(1-t) + q^n t} + \frac{q^n t}{p^n(1-t) + q^n t}\right) \\ &= \frac{p^n(1-t)}{p^n(1-t) + q^n t} \left(\frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \right) \\ &\quad + \frac{q^n t}{p^n(1-t) + q^n t} \left(\frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \right). \end{aligned}$$

Using 3.1 and 3.2, we have

$$b_{p,q}^{k,n}(t) = \left(\frac{p^k[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) b_{p,q}^{k,n+1}(t) + \left(1 - \frac{p^{k+1}[n-k]_{p,q}}{[n+1]_{p,q}}\right) b_{p,q}^{k+1,n+1}(t)$$

Theorem 3.2 *Each Lupaş (p, q) -analogue of the Bernstein function of degree n is a linear combination of two Lupaş (p, q) -analogues of the Bernstein functions of degree $n-1$:*

$$b_{p,q}^{k,n}(t) = \frac{q^{n-1} t}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k-1,n-1}(t) + \frac{p^{n-1}(1-t)}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k,n-1}(t) \quad (3.4)$$

$$b_{p,q}^{k,n}(t) = \frac{p^{n-k}q^{k-1}t}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k-1,n-1}(t) + \frac{p^{n-k-1}q^k(1-t)}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k,n-1}(t) \quad (3.5)$$

Proof We use the Pascal's type relations of the (p, q) -Binomial coefficient. According to formula 1.7,

$$b_{p,q}^{k,n}(t) = \frac{\left(p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}$$

or

$$\begin{aligned} b_{p,q}^{k,n}(t) &= \frac{p^{n-k}q^{k-1}t}{p^{n-1}(1-t) + q^{n-1}t} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{(k-1)(k-2)}{2}} t^{k-1} (1-t)^{n-k}}{\prod_{j=1}^{n-1} \{p^{j-1}(1-t) + q^{j-1}t\}} \\ &+ \frac{p^{n-k-1}q^k(1-t)}{p^{n-1}(1-t) + q^{n-1}t} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-1-k)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k-1}}{\prod_{j=1}^{n-1} \{p^{j-1}(1-t) + q^{j-1}t\}} \\ &= \frac{p^{n-1}q^{k-1}t}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k-1,n-1}(t) + \frac{p^{n-k-1}q^k(1-t)}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k,n-1}(t) \end{aligned}$$

or

$$\begin{aligned} b_{p,q}^{k,n}(t) &= \frac{\left(q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \\ &= \frac{q^{n-1}t}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k-1,n-1}(t) + \frac{p^{n-1}(1-t)}{p^{n-1}(1-t) + q^{n-1}t} b_{p,q}^{k,n-1}(t) \end{aligned}$$

4 Lupaş (p, q) -Bézier curves:

Let us define the Lupaş (p, q) -Bézier curves of degree n using the Lupaş (p, q) -analogues of the Bernstein functions as follows:

$$\mathbf{P}(t; p, q) = \sum_{i=0}^n \mathbf{P}_i b_{p,q}^{i,n}(t) \quad (4.1)$$

where $P_i \in R^3$ ($i = 0, 1, \dots, n$) and $p > q > 0$. P_i are control points. Joining up adjacent points P_i , $i = 0, 1, 2, \dots, n$ to obtain a polygon which is called the control polygon of Lupaş (p, q) -Bézier curves.

4.1 Some basic properties of Lupaş (p, q) -Bézier curves.

Theorem 4.1 *From the definition, we can derive some basic properties of Lupaş (p, q) -Bézier curves:*

1. Lupaş (p, q) -Bézier curves have geometric and affine invariance.
2. Lupaş (p, q) -Bézier curves lie inside the convex hull of its control polygon.
3. The end-point interpolation property: $\mathbf{P}(0; p, q) = \mathbf{P}_0$, $\mathbf{P}(1; p, q) = \mathbf{P}_n$.
4. (p, q) -inverse symmetry: the Lupaş (p, q) -Bézier curves obtained by reversing the order of the control points is the same as the Lupaş (p, q) -Bézier curves with q replaced by $\frac{1}{q}$ and p replaced by $\frac{1}{p}$.
5. Reducibility: when $p = 1$, formula 4.1 gives the q -Bézier curves.

Proof. These properties of Lupaş (p, q) -Bézier curves can be easily deduced from corresponding properties of the Lupaş (p, q) -analogue of the Bernstein functions. Here we only give the proof of property 4.

Let $\mathbf{P}_i^* = \mathbf{P}_{n-i}$, $i = 0, 1, \dots, n$, then we have

$$\begin{aligned} \mathbf{P}^*(t; p, q) &= \sum_{k=0}^n \mathbf{P}_i^* b_{p,q}^{k,n}(t) \\ &= \sum_{k=0}^n \mathbf{P}_i^* b_{\frac{1}{p}, \frac{1}{q}}^{k,n}(1-t) \\ &= \mathbf{P}(1-t; \frac{1}{p}, \frac{1}{q}). \end{aligned}$$

Theorem 4.2 *The end-point property of derivative:*

$$\mathbf{P}'(0; p, q) = \frac{[n]_{p,q}}{p^{n-1}} (\mathbf{P}_1 - \mathbf{P}_0)$$

$$\mathbf{P}'(1; p, q) = \frac{[n]_{p,q}}{q^{n-1}} (\mathbf{P}_n - \mathbf{P}_{n-1})$$

i.e. Lupaş (p, q) -Bézier curves are tangent to fore-and-aft edges of its control polygon at end points.

Proof: Let

$$\mathbf{P}(t; p, q) = \sum_{k=0}^n \mathbf{P}_k b_{p,q}^{k,n}(t) = \frac{\sum_{k=0}^n \mathbf{P}_k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \quad (4.2)$$

$$= \frac{\mathbf{V}(t; p, q)}{\mathbf{W}(t; p, q)} \quad (4.3)$$

or

$$\mathbf{P}(t; p, q) \mathbf{W}(t; p, q) = \mathbf{V}(t; p, q)$$

then on differentiating both hand side with respect to 't', we have

$$\mathbf{P}'(t; p, q) \mathbf{W}(t; p, q) + \mathbf{P}(t; p, q) \mathbf{W}'(t; p, q) = \mathbf{V}'(t; p, q).$$

Let

$$A_k^n(t; p, q) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k},$$

then

$$\mathbf{V}(t; p, q) = \sum_{k=0}^n \mathbf{P}_k A_k^n(t; p, q)$$

From property 2 of the Lupaş (p, q) -Bernstein functions, we obtain

$$\mathbf{W}(t; p, q) = \sum_{k=0}^n A_k^n(t; p, q)$$

as

$$\begin{aligned} (A_k^n(t; p, q))' &= \frac{[n]_{p,q}}{[k]_{p,q}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} k t^{k-1} (1-t)^{n-k} \\ &\quad - \frac{[n]_{p,q}}{[n-k]_{p,q}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} (n-k) t^k (1-t)^{n-k-1} \\ &= \frac{[n]_{p,q}}{[k]_{p,q}} q^{k-1} k A_{k-1}^{n-1}(t; p, q) - \frac{[n]_{p,q}}{[n-k]_{p,q}} p^{n-k-1} (n-k) A_k^{n-1}(t; p, q) \\ &= C_k^n A_{k-1}^{n-1}(t; p, q) - D_{n-k}^n A_k^{n-1}(t; p, q) \end{aligned}$$

where

$$C_k^n = \frac{[n]_{p,q}}{[k]_{p,q}} q^{k-1} k, \quad D_{n-k}^n = \frac{[n]_{p,q}}{[n-k]_{p,q}} p^{n-k-1} (n-k).$$

Then

$$\mathbf{V}(0; p, q) = \mathbf{P}_0 \mathbf{p}^{\frac{n(n-1)}{2}}, \quad \mathbf{W}(0; p, q) = p^{\frac{n(n-1)}{2}}$$

$$\mathbf{V}'(0; p, q) = (C_1^n \mathbf{P}_1 - D_n^n \mathbf{P}_0) p^{\frac{(n-1)(n-2)}{2}},$$

$$\mathbf{W}'(0; p, q) = (C_1^n - D_n^n) p^{\frac{(n-1)(n-2)}{2}},$$

hence

$$\mathbf{P}'(0; p, q) = \frac{[n]_{p,q}}{p^{n-1}} (\mathbf{P}_1 - \mathbf{P}_0)$$

Similarly, we have

$$\mathbf{V}(1; p, q) = \mathbf{P}_n \mathbf{q}^{\frac{n(n-1)}{2}}, \quad \mathbf{W}(1; p, q) = q^{\frac{n(n-1)}{2}}$$

$$\mathbf{V}'(1; p, q) = (C_n^n \mathbf{P}_n - D_1^n \mathbf{P}_{n-1}) q^{\frac{(n-1)(n-2)}{2}},$$

$$\mathbf{W}'(1; p, q) = (C_n^n - D_1^n) q^{\frac{(n-1)(n-2)}{2}},$$

hence

$$\mathbf{P}'(1; p, q) = \frac{[n]_{p,q}}{q^{n-1}} (\mathbf{P}_n - \mathbf{P}_{n-1})$$

Theorem 4.3 *Planar Lupaş (p, q) -Bézier curves are variation diminishing, which means that the number of times any straight line crosses the Lupaş (p, q) -Bézier curve is no more than the number of times it crosses the control polygon.*

Proof. For any polynomial $f(t)$, we denote $Z_{t \in I \subseteq (0, \infty)}[f(t)]$ as the number of positive roots of $f(t)$ on the interval I . For any vector $V = (v_0, v_1, \dots, v_n)$, we write $S^-(v_0, v_1, \dots, v_n)$ to denote the number of strict sign changes in the vector V .

Because the sequence of functions $(1, t, \dots, t^n)$ is totally positive on $[0, 1]$, then for any sequence of real numbers a_0, a_1, \dots, a_n , $Z_{0 < t < 1}[a_0 + a_1 t + \dots + a_n t^n] = S^-(a_0 + a_1 t + \dots + a_n t^n) \leq S^-(a_0, a_1, \dots, a_n)$.

Let C denote a planar Lupaş (p, q) -Bézier curve, L any straight line, and let $I(C, L)$ the number of times C crosses L . Establish the rectangular coordinate system whose abscissa axis is L . Because Lupaş (p, q) -Bézier curves are geometric invariant, we can denote $(x_i, y_i) (i = 0, 1, \dots, n)$ the new coordinates of the control points. Let P denote the control polygon and $I(P, L)$ the number of times P crosses L . Then we will prove that $I(C, L) \leq I(P, L)$.

We make a parameter transformation. Let $u = \frac{t}{1-t}$, $t \in (0, 1)$, so that $u \in (0, \infty)$. Then

$$\begin{aligned} I(C, L) &= Z_{0 < t < 1} \left[\sum_{k=0}^n \mathbf{y}_k b_{p,q}^{k,n}(t) \right] \\ &= Z_{0 < t < 1} \left[\frac{\sum_{k=0}^n \mathbf{y}_k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \right] \\ &= Z_{0 < u < \infty} \left[\frac{\sum_{k=0}^n \mathbf{y}_k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=1}^n \{p^{j-1} + q^{j-1}u\}} \right] \\ &= Z_{0 < u < \infty} \left[\sum_{k=0}^n \mathbf{y}_k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} u^k \right] \\ &\leq S^-\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} y_0, \begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q} y_1, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} y_n \right) \\ &= S^-(y_0, y_1, \dots, y_n) \end{aligned}$$

From 4.3, we can obtain the following two corollaries:

Corollary 4.4 *Convexity-preserving: the planar Lupaş (p, q) -Bézier curve is convex, as long as its control polygon is convex.*

Corollary 4.5 *Monotonicity-preserving: let the control polygon be monotonically increasing (decreasing) in a given direction, then the planar Lupaş (p, q) -Bézier curve is also monotonically increasing (decreasing).*

4.2 Degree elevation for Lupaş (p, q) -Bézier curves

Lupaş (p, q) -Bézier curves have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. Using the technique of degree elevation, we can increase the flexibility of a given curve.

$$\mathbf{P}(t; p, q) = \sum_{k=0}^n \mathbf{P}_k b_{p,q}^{k,n}(t)$$

$$\mathbf{P}(t; p, q) = \sum_{k=0}^{n+1} \mathbf{P}_k^* b_{p,q}^{k,n+1}(t),$$

where

$$\mathbf{P}_k^* = \left(1 - \frac{p^k [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_{k-1} + \left(\frac{p^k [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_k \quad (4.4)$$

The statement above can be derived using the identities (3.1) and (3.2). Consider

$$\mathbf{P}(t; p, q) = \frac{p^n(1-t)}{p^n(1-t) + q^n t} \mathbf{P}(t; p, q) + \frac{q^n t}{p^n(1-t) + q^n t} \mathbf{P}(t; p, q)$$

We obtain

$$\mathbf{P}(t; p, q) = \sum_{k=0}^n \left(p^k \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_k^0 b_{p,q}^{k,n+1}(t) + \sum_{k=0}^n \left(1 - \frac{p^{k+1} [n-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_{k-1}^0 b_{p,q}^{k+1,n+1}(t)$$

Now by shifting the limits, we have

$$\mathbf{P}(t; p, q) = \sum_{k=0}^{n+1} \left(p^k \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_k^0 b_{p,q}^{k,n+1}(t) + \sum_{k=0}^{n+1} \left(1 - \frac{p^k [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_{k-1}^0 b_{p,q}^{k,n+1}(t)$$

where \mathbf{P}_{-1}^0 is defined as the zero vector. Comparing coefficients on both side, we have

$$\mathbf{P}_k^* = \left(1 - \frac{p^k [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_{k-1} + \left(\frac{p^k [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P}_k$$

where $k = 0, 1, 2, \dots, n+1$ and $\mathbf{P}_{-1} = \mathbf{P}_{n+1} = 0$.

When $p = 1$, formula 4.4 reduce to the degree evaluation formula of the q -Bézier curves. If we let $P = (P_0, P_1, \dots, P_n)^T$ denote the vector of control points of the initial Lupaş (p, q) -Bézier curve of degree n , and $\mathbf{P}^{(1)} = (P_0^*, P_1^*, \dots, P_{n+1}^*)^T$ the vector of control points of the degree elevated Lupaş (p, q) -Bézier curve of degree $n+1$, then we can represent the degree elevation procedure as:

$$\mathbf{P}^{(1)} = T_{n+1} \mathbf{P}$$

where

$$T_{n+1} = \frac{1}{[n+1]_{p,q}} \begin{bmatrix} [n+1]_{p,q} & 0 & \dots & 0 & 0 \\ [n+1]_{p,q} - p[n]_{p,q} & p[n]_{p,q} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & [n+1]_{p,q} - p^{n-1}[2]_{p,q} & p^{n-1}[2]_{p,q} & 0 \\ 0 & 0 & \dots & [n+1]_{p,q} - p^n[1]_{p,q} & p^n[1]_{p,q} \\ 0 & 0 & \dots & 0 & [n+1]_{p,q} \end{bmatrix}_{(n+2) \times (n+1)}$$

For any $l \in \mathbb{N}$, the vector of control points of the degree elevated Lupaş (p, q) -Bézier curves of degree $n + l$ is: $\mathbf{P}^{(l)} = T_{n+l} T_{n+2} \dots T_{n+1} \mathbf{P}$. As $l \rightarrow \infty$, the control polygon $\mathbf{P}^{(l)}$ converges to a Lupaş (p, q) -Bézier curve.

5 Construction of (p, q) -analogue of Lupaş operators and its limit form

In this section, we present (p, q) -analogue of Lupaş Bernstein operators and its limit form as follows:

The linear operators $L_{p,q}^n : C[0, 1] \rightarrow C[0, 1]$

$$L_{p,q}^n(f; x) = \sum_{k=0}^n \frac{f\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}}\right) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-x) + q^{j-1}x\}}, \quad (5.1)$$

is (p, q) -analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş (p, q) -Bernstein operators turns out to be Lupaş q -Bernstein operators as given in [9, 19].

It follows directly from the definition that operators $L_{p,q}^n(f, t)$ posses the end point interpolation property, that is

$$L_{p,q}^n(f, 0) = f(0), \quad L_{p,q}^n(f, 1) = f(1)$$

for all $p > q > 0$ and all $n = 1, 2, \dots$.

Now we show that (p, q) -analogue of Lupaş operator reproduces linear and constant functions.

By some simple computation, we have

Some auxillary results:

$$(1) L_{p,q}^n(1, \frac{u}{u+1}) = 1$$

$$(2) L_{p,q}^n(t, \frac{u}{u+1}) = \frac{u}{u+1}$$

$$(3) L_{p,q}^n(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \left(\frac{qu}{p+qu} \right) \frac{[n-1]_{p,q}}{[n]_{p,q}}$$

or equivalently for $x = \frac{u}{u+1}$

$$(1) L_{p,q}^n(1, x) = 1$$

$$(2) L_{p,q}^n(t, x) = x$$

$$(3) L_{p,q}^n(t^2, x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2 x^2}{p(1-x)+qx} \frac{[n-1]_{p,q}}{[n]_{p,q}}$$

Proof: (1) is obvious using (p, q) -analogue of Lupas Bernstein basis function.

$$L_{p,q}^n(t; \frac{u}{u+1}) = \sum_{k=0}^n \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} = 1,$$

(2)

$$\begin{aligned} L_{p,q}^n(t; \frac{u}{u+1}) &= \sum_{k=0}^n \frac{\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} \right) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \\ &= \sum_{k=0}^n \frac{\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} \right) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=0}^{n-1} \{p^j(1-t) + q^j t\}} \\ &= \sum_{k=0}^n \frac{p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=0}^{n-1} \{p^j(1-t) + q^j t\}} \\ &= \sum_{k=1}^n \frac{p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=0}^{n-1} \{p^j + q^j u\}} \end{aligned}$$

$$\begin{aligned}
\Rightarrow L_{p,q}^n(t; \frac{u}{u+1}) &= \sum_{k=0}^{n-1} \frac{p^{n-k-1} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} u^{k+1}}{\prod_{j=0}^{n-1} \{p^j + q^j u\}} \\
&= \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{p^{n-1} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j p + q^j(qu)\}} \\
&= \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{\left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p}u)\}} \\
&= \frac{u}{u+1}
\end{aligned}$$

or equivalently for $x = \frac{u}{u+1}$

$$L_{p,q}^n(t, x) = x.$$

$$(3) \text{ To prove, } L_{p,q}^n(t^2, x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2 x^2}{p(1-x)+qx} \frac{[n-1]_{p,q}}{[n]_{p,q}}.$$

Consider

$$\begin{aligned}
L_{p,q}^n(t^2; \frac{u}{u+1}) &= \sum_{k=0}^n \frac{\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} \right)^2 \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=0}^{n-1} \{p^j + q^j u\}} \\
&= \sum_{k=0}^n \frac{p^{2n-2k} \frac{[k]_{p,q}}{[n]_{p,q}} \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=0}^{n-1} \{p^j + q^j u\}} \\
&= \frac{u}{u+1} \sum_{k=0}^{n-1} p^{n-1-k} \frac{[k+1]_{p,q}}{[n]_{p,q}} \frac{\left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p}u)\}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow L_{p,q}^n(t^2; \frac{u}{u+1}) &= \frac{u}{u+1} \frac{[n-1]_{p,q}}{[n]_{p,q}} \sum_{k=0}^{n-1} p^{n-1-k} \left(\frac{p^k + q[k]_{p,q}}{[n-1]_{p,q}} \right) \frac{\left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p} \right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j (\frac{qu}{p})\}} \\
&= \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \frac{[n-1]_{p,q}}{[n]_{p,q}} \sum_{k=0}^{n-1} p^{n-k-1} \frac{\left[\begin{matrix} n-2 \\ k-1 \end{matrix} \right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p} \right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j (\frac{qu}{p})\}} \\
&= \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \frac{[n-1]_{p,q}}{[n]_{p,q}} \frac{\frac{qu}{p}}{1 + \frac{qu}{p}} \sum_{k=0}^{n-2} p^{n-2} \frac{\left[\begin{matrix} n-2 \\ k \end{matrix} \right]_{p,q} p^{\frac{(n-k-2)(n-k-3)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{q^2 u}{p^2} \right)^k}{p^{n-2} \prod_{j=0}^{n-3} \{p^j + q^j (\frac{q^2 u}{p^2})\}} \\
&= \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \left(\frac{qu}{p+qu} \right) \frac{[n-1]_{p,q}}{[n]_{p,q}}
\end{aligned}$$

Theorem 5.1 Let $0 < q_n < p_n \leq 1$ such that $\lim_{n \rightarrow \infty} p_n = 1$, $\lim_{n \rightarrow \infty} q_n = 1$ and $\lim_{n \rightarrow \infty} p_n^n = a$, $\lim_{n \rightarrow \infty} q_n^n = b$ with $0 < a, b \leq 1$. Then for each $f \in C[0, 1]$, $L_{p,q}^n(f; x)$ converges uniformly to f on $C[0, 1]$.

Proof: Proof is obvious using the following Korovkin's theorem.

Let (T_n) be a sequence of positive linear operators from $\mathcal{C}[a, b]$ into $\mathcal{C}[a, b]$. Then $\lim_n \|T_n(f, x) - f(x)\|_{\mathcal{C}[a, b]} = 0$, for all $f \in \mathcal{C}[a, b]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_{\mathcal{C}[a, b]} = 0$, for $i = 0, 1, 2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

Remark:

For $q \in (0, 1)$ and $p \in (q, 1]$ it is obvious that $\lim_{n \rightarrow \infty} [n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to reach to convergence results of the operator $L_{p,q}^n(f; x)$, we take a sequence $q_n \in (0, 1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n \rightarrow \infty} p_n = 1$, $\lim_{n \rightarrow \infty} q_n = 1$ and $\lim_{n \rightarrow \infty} p_n^n = a$, $\lim_{n \rightarrow \infty} q_n^n = b$ with $0 < a, b \leq 1$. So we get $\lim_{n \rightarrow \infty} [n]_{p_n, q_n} = \infty$.

(p, q)-analogue of the limit Lupaş Bernstein operators

First, let $0 < q < p < 1$ and $\lim_n \frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} = 1 - \left(\frac{q}{p}\right)^k$

We define linear operators $L_{p,q}^\infty$ defined on $C[0, 1]$ as

$$L_{p,q}^\infty(f; x) = \begin{cases} \sum_{k=0}^{\infty} f(1 - (\frac{q}{p})^k) b_{p,q}^{k,\infty}, & \text{if } x \in [0, 1) \\ f(1), & \text{if } x = 1, \end{cases} \quad (5.2)$$

which is (p, q) -analogue of the limit Lupaş Bernstein operators.

For more details on Limit Lupaş q -analogue of Bernstein operators, one can refer [9, 19].

Note that the function $L_{p,q}^\infty(f; x)$ is well-defined on $[0, 1]$ whenever $f(x)$ is bounded on $[0, 1]$.

Theorem 5.2 Let $f \in C[0, 1]$, $g(x) = f(1 - x)$. Then for any $p > q > 0$,

$$L_{p,q}^n(f; t) = L_{\frac{1}{p}, \frac{1}{q}}^n(g; 1 - t), \quad \text{for } t \in [0, 1]$$

Proof:

The proof of above theorem follow easily along the lines of [19] and using the following relations,

$$[n]_{p,q} = (pq)^{n-1} [n]_{\frac{1}{q}, \frac{1}{p}},$$

$$\frac{p^k [n-k]_{p,q}}{[n]_{p,q}} = 1 - \frac{p^{k-n} [k]_{\frac{1}{q}, \frac{1}{p}}}{[n]_{\frac{1}{q}, \frac{1}{p}}}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}, \frac{1}{p}} \frac{(pq)^{\frac{k(2n-1-k)}{2}}}{(pq)^{\frac{k(k-1)}{2}}}.$$

Remark: For $p = 1$, this equality coincides with formula (10), Theorem 3 in [19].

Corollary 1: Let $p \neq 1$ be fixed, $f \in C[0, 1]$, and $g(x) = f(1-x)$. Then, for $x \in [0, 1]$, $L_{p,q}^n(f; t)$ converges uniformly to $L_{p,q}^\infty(f; t)$ for any $p > 0$ being fixed where

$$L_{p,q}^\infty(f; t) = \begin{cases} L_{p,q}^\infty(f; t), & \text{if } 0 < q < p < 1 \\ L_{\frac{1}{q}, \frac{1}{p}}^\infty(f; t), & \text{if } p > q > 1 \end{cases} \quad (5.3)$$

An explicit form of the limit function for $0 < q < p < 1$ is given by 5.2.

Theorem 5.3 Let $p > q > 0$, $p \neq 1$ be fixed and $f \in C[0, 1]$. Then $L_{p,q}^\infty(f; t) = f(t)$ for all $t \in [0, 1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Corollary 2: Operators $L_{p,q}^\infty(f; x)$ reproduces linear functions, that is $L_{p,q}^\infty(at + b; x) = ax + b$ for all $p > q > 0$ and all $n = 1, 2, \dots$ where $a, b \in \mathbb{R}$.

Note: (p, q) -analogue of Lupas Bernstein operator have an advantage of generating positive linear operators for all $p > q > 0$, whereas (p, q) -analogue of Bernstein polynomials introduced by Mursaleen et al. generate positive linear operators only if $0 < q < p < 1$.

6 (p, q) -de Casteljau algorithm for Lupas Bézier curves :

Lupas (p, q) -Bézier curves of degree n can be written as two kinds of linear combination of two Lupas (p, q) -Bézier curves of degree $n - 1$, and we can get the two selectable algorithms to evaluate Lupas (p, q) -Bézier curves. The algorithms can be expressed as:

Algorithm 1.

$$\begin{cases} \mathbf{P}_i^0(t; p, q) \equiv \mathbf{P}_i^0 \equiv \mathbf{P}_i & i = 0, 1, 2, \dots, n \\ \mathbf{P}_i^r(t; p, q) = \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} \mathbf{P}_{i+1}^{r-1}(t; p, q) + \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} \mathbf{P}_i^{r-1}(t; p, q) \\ r = 1, \dots, n, \quad i = 0, 1, 2, \dots, n-r, \end{cases} \quad (6.1)$$

or

$$\begin{cases} \mathbf{P}_i^0(t; p, q) \equiv \mathbf{P}_i^0 \equiv \mathbf{P}_i & i = 0, 1, 2, \dots, n \\ \mathbf{P}_i^r(t; p, q) = \frac{p^{n-i-r} q^i t}{p^{n-r}(1-t) + q^{n-r} t} \mathbf{P}_{i+1}^{r-1}(t; p, q) + \frac{p^{n-i-r} q^i (1-t)}{p^{n-r}(1-t) + q^{n-r} t} \mathbf{P}_i^{r-1}(t; p, q) \\ r = 1, \dots, n, \quad i = 0, 1, 2, \dots, n-r, \end{cases} \quad (6.2)$$

Then

$$\mathbf{P}(t; p, q) = \sum_{i=0}^{n-1} \mathbf{P}_i^1(t; p, q) = \dots = \sum \mathbf{P}_i^r(t; p, q) b_{p,q}^{i,n-r}(t) = \dots = \mathbf{P}_0^n(t; p, q) \quad (6.3)$$

It is clear that the results can be obtained from Theorem (3.2). When $p = 1$, formula (6.1) and (6.2) recover the de Casteljau algorithms of classical q -Bézier curves. Let $P^0 = (P_0, P_1, \dots, P_n)^T$, $P^r = (P_0^r, P_1^r, \dots, P_{n-r}^r)^T$, then de Casteljau algorithm can be expressed as:

Algorithm 2.

$$\mathbf{P}^r(t; p, q) = M_r(t; p, q) \dots M_2(t; p, q) M_1(t; p, q) \mathbf{P}^0 \quad (6.4)$$

where $M_r(t; p, q)$ is a $(n-r+1) \times (n-r+2)$ matrix and

$$M_r(t; p, q) = \begin{bmatrix} \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} & \dots & 0 & 0 \\ 0 & \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} & 0 \\ 0 & 0 & \dots & \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} \end{bmatrix}$$

or

$$M_r(t; p, q) = \begin{bmatrix} \frac{p^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{p^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} & \dots & 0 & 0 \\ 0 & \frac{p^{n-r-1} q(1-t)}{(1-t) + q^{n-r} t} & \frac{p^{n-r-1} q t}{p^{n-r}(1-t) + q^{n-r} t} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{p q^{n-r-1}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{p q^{n-r-1} t}{p^{n-r}(1-t) + q^{n-r} t} & 0 \\ 0 & 0 & \dots & \frac{q^{n-r}(1-t)}{p^{n-r}(1-t) + q^{n-r} t} & \frac{q^{n-r} t}{p^{n-r}(1-t) + q^{n-r} t} \end{bmatrix}$$

7 Tensor product Lupaş (p, q) -Bézier surfaces on $[0, 1] \times [0, 1]$

We define a two-parameter family $\mathbf{P}(u, v)$ of tensor product surfaces of degree $m \times n$ as follow:

$$\mathbf{P}(u, v) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{P}_{i,j} b_{p_1, q_1}^{i,m}(u) b_{p_2, q_2}^{j,n}(v), \quad (u, v) \in [0, 1] \times [0, 1], \quad (7.1)$$

where $\mathbf{P}_{i,j} \in \mathbb{R}^3$ ($i = 0, 1, \dots, m, j = 0, 1, \dots, n$) and two real numbers $p_1 > q_1 > 0$, $p_2 > q_2 > 0$, $b_{p_1, q_1}^{i,m}(u)$, $b_{p_2, q_2}^{j,n}(v)$ are Lupaş (p, q) -analogue of Bernstein functions respectively with the parameter p_1, q_1 and p_2, q_2 . We call the parameter surface tensor product Lupaş (p, q) -Bézier surface with degree

$m \times n$. We refer to the $\mathbf{P}_{i,j}$ as the control points. By joining up adjacent points in the same row or column to obtain a net which is called the control net of tensor product Lupaş (p, q) -Bézier surface.

7.1 Properties

1. **Geometric invariance and affine invariance property:** Since

$$\sum_{i=0}^m \sum_{j=0}^n b_{p_1, q_1}^{i, m}(u) b_{p_2, q_2}^{j, n}(v) = 1, \quad (7.2)$$

$\mathbf{P}(u, v)$ is an affine combination of its control points.

2. **Convex hull property:** $\mathbf{P}(u, v)$ is a convex combination of $\mathbf{P}_{i,j}$ and lies in the convex hull of its control net.

3. **Isoparametric curves property:** The isoparametric curves $v = v^*$ and $u = u^*$ of a tensor product Lupaş (p, q) -Bézier surface are respectively the Lupaş (p, q) -Bézier curves of degree m and degree n , namely,

$$\mathbf{P}(u, v^*) = \sum_{i=0}^m \left(\sum_{j=0}^n \mathbf{P}_{i,j} b_{p_2, q_2}^{j, n}(v^*) \right) b_{p_1, q_1}^{i, m}(u) \quad , \quad u \in [0, 1];$$

$$\mathbf{P}(u^*, v) = \sum_{j=0}^n \left(\sum_{i=0}^m \mathbf{P}_{i,j} b_{p_1, q_1}^{i, m}(u^*) \right) b_{p_2, q_2}^{j, n}(v) \quad , \quad v \in [0, 1]$$

The boundary curves of $\mathbf{P}(u, v)$ are evaluated by $\mathbf{P}(u, 0)$, $\mathbf{P}(u, 1)$, $\mathbf{P}(0, v)$ and $\mathbf{P}(1, v)$.

4. **Corner point interpolation property:** The corner control net coincide with the four corners of the surface. Namely, $\mathbf{P}(0, 0) = \mathbf{P}_{0,0}$, $\mathbf{P}(0, 1) = \mathbf{P}_{0,n}$, $\mathbf{P}(1, 0) = \mathbf{P}_{m,0}$, $\mathbf{P}(1, 1) = \mathbf{P}_{m,n}$,

5. **Reducibility:** When $p_1 = p_2 = 1$, formula (7.1) reduces to a tensor product q -Bézier patch.

7.2 Degree elevation and (p, q) -de Casteljau algorithm for Lupaş Bézier surface

Let $\mathbf{P}(u, v)$ be a tensor product Lupaş (p, q) -Bézier surface of degree $m \times n$. As an example, let us take obtaining the same surface as a surface of degree $(m+1) \times (n+1)$. Hence we need to find new control points $\mathbf{P}_{i,j}^*$ such that

$$\mathbf{P}(u, v) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{P}_{i,j} b_{p_1, q_1}^{i, m}(u) b_{p_2, q_2}^{j, n}(v) = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} \mathbf{P}_{i,j}^* b_{p_1, q_1}^{i, m+1}(u) b_{p_2, q_2}^{j, n+1}(v) \quad (7.3)$$

$$\text{Let } \alpha_i = 1 - \frac{p_1^i [m+1-i]_{p_1, q_1}}{[m+1]_{p_1, q_1}}, \quad \beta_j = 1 - \frac{p_2^j [n+1-j]_{p_2, q_2}}{[n+1]_{p_2, q_2}}.$$

Then

$$\mathbf{P}_{i,j}^* = \alpha_i \beta_j \mathbf{P}_{i-1, j-1} + \alpha_i (1 - \beta_j) \mathbf{P}_{i-1, j} + (1 - \alpha_i) \beta_j \mathbf{P}_{i, j-1} + (1 - \alpha_i) (1 - \beta_j) \mathbf{P}_{i, j} \quad (7.4)$$

which can be written in matrix form as

$$\begin{bmatrix} 1 - \frac{p_1^i [m+1-i]_{p_1, q_1}}{[m+1]_{p_1, q_1}} & \frac{p_1^i [m+1-i]_{p_1, q_1}}{[m+1]_{p_1, q_1}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{i-1, j-1} & \mathbf{P}_{i-1, j} \\ \mathbf{P}_{i, j-1} & \mathbf{P}_{i, j} \end{bmatrix} \begin{bmatrix} 1 - \frac{p_2^j [n+1-j]_{p_2, q_2}}{[n+1]_{p_2, q_2}} \\ \frac{p_2^j [n+1-j]_{p_2, q_2}}{[n+1]_{p_2, q_2}} \end{bmatrix}$$

The de Casteljau algorithms are also easily extended to evaluate points on a Lupaş (p, q) -Bézier surface. Given the control net $\mathbf{P}_{i,j} \in \mathbb{R}^3, i = 0, 1, \dots, m, j = 0, 1, \dots, n$.

$$\left\{ \begin{array}{l} \mathbf{P}_{i,j}^{0,0}(u, v) \equiv \mathbf{P}_{i,j}^{0,0} \equiv \mathbf{P}_{i,j} \quad i = 0, 1, 2, \dots, m; \quad j = 0, 1, 2, \dots, n. \\ \mathbf{P}_{i,j}^{r,r}(u, v) = \left[\frac{p_1^{m-r}(1-u)}{p_1^{m-r}(1-u)+q_1^{m-r}u} \quad \frac{q_1^{m-r}u}{p_1^{m-r}(1-u)+q_1^{m-r}u} \right] \begin{bmatrix} \mathbf{P}_{i,j}^{r-1,r-1} & \mathbf{P}_{i,j+1}^{r-1,r-1} \\ \mathbf{P}_{i+1,j}^{r-1,r-1} & \mathbf{P}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \left[\frac{\frac{p_2^{n-r}(1-v)}{p_2^{n-r}(1-v)+q_2^{n-r}v}}{\frac{q_2^{n-r}v}{p_2^{n-r}(1-v)+q_2^{n-r}v}} \right] \\ r = 1, \dots, k, \quad k = \min(m, n) \quad i = 0, 1, 2, \dots, m-r; \quad j = 0, 1, \dots, n-r \end{array} \right. \quad (7.5)$$

or

$$\left\{ \begin{array}{l} \mathbf{P}_{i,j}^{0,0}(u, v) \equiv \mathbf{P}_{i,j}^{0,0} \equiv \mathbf{P}_{i,j} \quad i = 0, 1, 2, \dots, m; \quad j = 0, 1, 2, \dots, n. \\ \mathbf{P}_{i,j}^{r,r}(u, v) = \left[\frac{p_1^{m-i-r}q_1^i(1-u)}{p_1^{m-i-r}(1-u)+q_1^{m-i-r}u} \quad \frac{p_1^{m-i-r}q_1^i u}{p_1^{m-i-r}(1-u)+q_1^{m-i-r}u} \right] \begin{bmatrix} \mathbf{P}_{i,j}^{r-1,r-1} & \mathbf{P}_{i,j+1}^{r-1,r-1} \\ \mathbf{P}_{i+1,j}^{r-1,r-1} & \mathbf{P}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \left[\frac{\frac{p_2^{n-j-r}q_2^j(1-v)}{p_2^{n-j-r}(1-v)+q_2^{n-j-r}v}}{\frac{p_2^{n-j-r}q_2^j v}{p_2^{n-j-r}(1-v)+q_2^{n-j-r}v}} \right] \\ r = 1, \dots, k, \quad k = \min(m, n) \quad i = 0, 1, 2, \dots, m-r; \quad j = 0, 1, \dots, n-r \end{array} \right. \quad (7.6)$$

When $m = n$, one can directly use the algorithms above to get a point on the surface. When $m \neq n$, to get a point on the surface after k applications of formula (7.5) or (7.6), we perform formula (6.4) for the intermediate point $\mathbf{P}_{i,j}^{k,k}$.

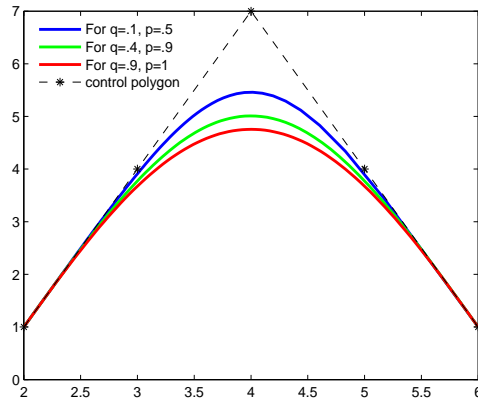
Note: We get Lupaş q -Bézier curves and surfaces for $(u, v) \in [0, 1] \times [0, 1]$ when we set the parameter $p_1 = p_2 = 1$ as proved in [4].

8 Shape control of (p, q) -Bézier curves and surfaces

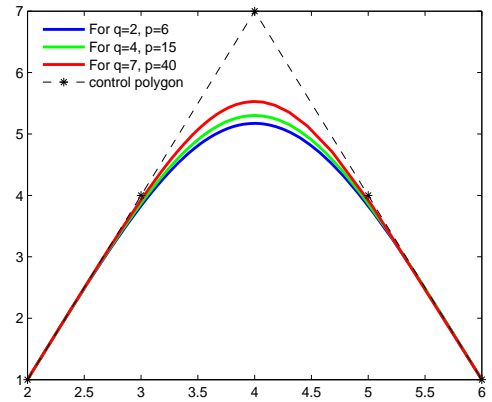
We have constructed Lupaş type (p, q) -Bernstein functions which holds the end point interpolation property as shown in following figures of curves and surfaces. It can be also observed that the curves (surfaces) generated is contained within the convex hull of the control polygon (control net) for different values of p and q .

Parameter p and q has been used to control the shape of curves and surfaces: if $0 < q < p \leq 1$, as p and q decreases, the curves (surfaces) move close to the control polygon (control net), as p and q increases, the curves (surfaces) move away from the control polygon (control net).

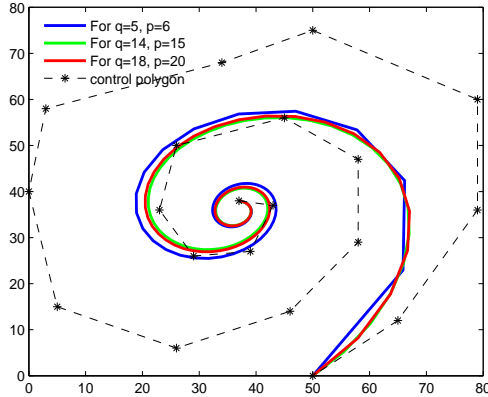
If $1 < q < p$, the effects of p and q are opposite, as p and q decreases, the curves (surfaces) move away from the control polygon (control net), as p and q increases, the curves (surfaces) move close to the control polygon (control net) which can be seen in the following figures.



(a) $0 < q < p \leq 1$

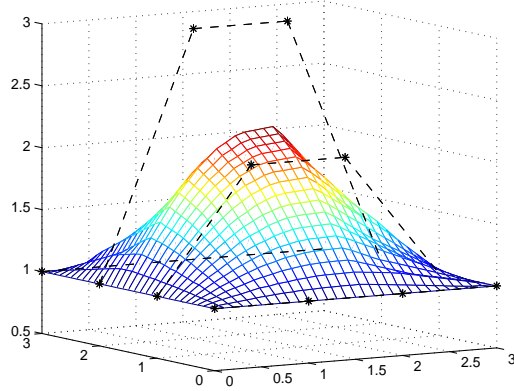


(b) $1 < q < p$

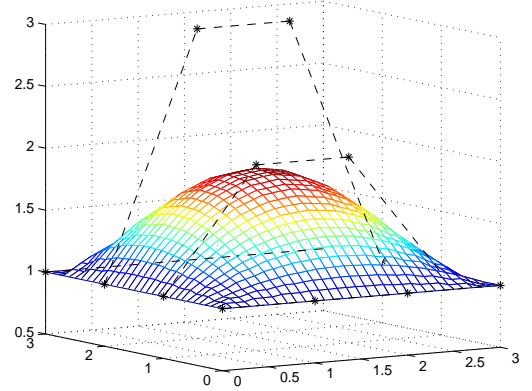


(c) $1 < q < p$

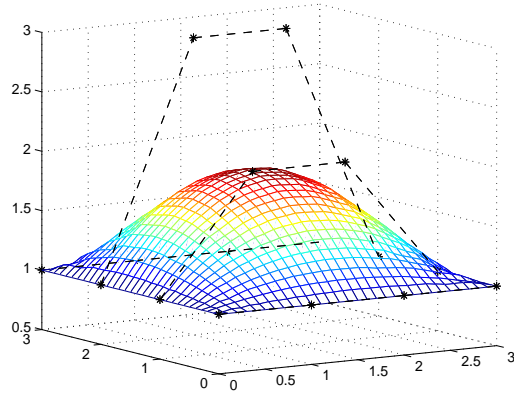
Figure 2: The effect of the shape of Lupaş (p, q) -Bézier curves



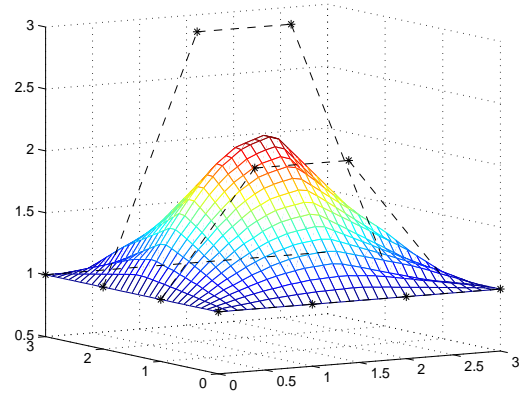
(a) $q_1 = q_2 = .1, p_1 = p_2 = .7$



(b) $q_1 = q_2 = .99, p_1 = p_2 = 1$



(c) $q_1 = q_2 = 3, p_1 = p_2 = 5$



(d) $q_1 = q_2 = 20, p_1 = p_2 = 200$

Figure 3: The effect of the shape of Lupaş (p, q) -Bézier surfaces

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